Continua of localized wave solutions via a complex similarity transformation

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In the following, we obtain continua of localized wave solutions to the scalar homogeneous wave, damped wave, and Klein-Gordon equations. We do this by utilizing the fact that similar *Ansätze* (all of which involve a free-particle time-dependent Schrödinger-like equation) may be used to satisfy all three of these partial differential equations. This Schrödinger-like equation is reduced to an ordinary differential equation (ODE) using a dimensionless complex similarity transformation. A general solution to this ODE involving confluent hypergeometric functions is found. For an azimuthal dependence $\exp(i\nu\phi), \nu\in\mathbb{R}$, this general solution includes many of the previously determined localized wave solutions as special cases.

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I. INTRODUCTION

Over the past ten years, large numbers of new interesting packetlike macroscopic solutions of Maxwell's equations [1–4], the scalar homogeneous wave equation [2,5–7], the Klein-Gordon equation [7,8], and other important partial differential equations of mathematical physics have been discovered. These solutions are exact for nondispersive media and describe localized wave propagation in space-time. Such localized wave functions are characterized as having shapes and/or amplitudes that can be maintained over much larger distances than their traditional monochromatic continuous-wave counterparts. While individual localized waves share with the plane wave the property of having infinite energy, appropriate superpositions of fundamental localized waves have resulted in wave forms having finite energy, thus leading to the possibility of launching fields with extended localization properties from simple antennas or antenna arrays.

In this paper we obtain continua of localized wave solutions to the scalar homogeneous wave, damped wave, and Klein-Gordon equations. We do this by utilizing the fact that similar *Ansätze* (all of which involve a free-particle time-dependent Schrödinger-like equation) may be used to satisfy all three of the above partial differential equations. Thus, solutions of this Schrödinger-like equation take on greater significance than they had previously. We have found that this equation can be reduced to an ordinary differential equation via a specific complex similarity transformation. A general solution of this ordinary differential equation has been determined, and for $\exp(i\nu\phi)$, $\nu\in\mathbb{R}$, azimuthal dependence, this general solution includes many of the previously determined localized wave solutions as special cases.

In Sec. II, we discuss the *Ansätze* for the wave, damped wave, and Klein-Gordon equations and show how they can be used on two previously known examples of localized waves for the scalar wave equation. These examples are the fundamental Gaussian solution [8,9] and the family of Bessel-Gauss pulses [10]. In Sec. III, we solve the free-particle time-dependent Schrödinger-like equation for the azimuthally symmetric case using a complex similarity transformation. In Sec. IV the same transformation is used to solve the case of general azimuthal dependence. Section V contains our conclusions.

II. *ANSÄTZE* FOR THE DAMPED WAVE AND KLEIN-GORDON EQUATIONS

It has been known for some time that localized wave solutions of the homogeneous wave equation can be found by transforming the wave equation variables z and t to the characteristic variables $\zeta=z-ct$, $\eta=z+ct$ [1–7]. This causes the homogeneous wave equation, $\Box(\mathbf{r},t)=0$, to have the form

$$\left[\nabla_t^2+4\frac{\partial^2}{\partial\zeta\partial\eta}\right]\psi(\mathbf{r},t)=0\ , \tag{1}$$

where $\nabla_t^2=\nabla^2-\partial^2/\partial z^2$. Using cylindrical coordinates and making the further assumption that

$$\psi(\mathbf{r},t)=f(\rho,\phi,z-ct)\exp[i\beta(z+ct)]\ , \tag{2}$$

forces the function $f(\rho,\phi,z-ct)$ to satisfy a free-particle time-dependent Schrödinger-like equation,

$$\left[\nabla_t^2+4i\beta\frac{\partial}{\partial\zeta}\right]f(\rho,\phi,\zeta)=0\ . \tag{3}$$

Thus solutions to (3) take on a new significance since any $f(\rho,\phi,\zeta)$ will provide a new solution to the homogeneous wave equation when the form (2) is used.

We use the fact that the general form [8]

$$\Phi(\mathbf{r},t)=f(\rho,\phi,z-ct)\exp[-|b|ct+i(zp_1+ctp_2)]\ , \tag{4}$$

where

$$b=\frac{-\mu\sigma c}{2},\quad p_1=\beta+b^2/4\beta,\quad p_2=\beta-b^2/4\beta \tag{5}$$

satisfies the homogeneous damped wave equation,

$$\left[\nabla^2-\epsilon\mu\frac{\partial^2}{\partial t^2}-\mu\sigma\frac{\partial}{\partial t}\right]\Phi(\mathbf{r},t)=0 \tag{6}$$

where ϵ is the permittivity, μ is the permeability, and σ is the conductivity, provided the function f in (4) satisfies (3).

Similarly [8],

$$\pi(\mathbf{r},t)=f(\rho,\phi,z-ct)\exp[i(zp_3+ctp_4)] , \tag{7}$$

where

$$p_3=\beta-\bar{\mu}^2/4\beta, \quad p_4=\beta+\bar{\mu}^2/4\beta \tag{8}$$

satisfies the Klein-Gordon equation,

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\bar{\mu}^2\right]\pi(\mathbf{r},t)=0 , \tag{9}$$

provided f in (7) again satisfies (3). These two forms, (4) and (7), can be verified by direct substitution. Solutions to (3) now take on even greater importance since they can provide solutions to the damped wave and Klein-Gordon equations as well as the wave equation.

Two important examples that can be used in the general forms given by (2), (4), and (7) are the fundamental Gaussian localized wave [8,9] and the family of Bessel-Gauss pulses [10,11]. The original fundamental Gaussian function was a localized solution of the homogeneous wave equation, but recently Donnelly and Ziolkowski have obtained fundamental Gaussian localized solutions to the damped wave and Klein-Gordon equations also. It follows immediately from their work and (4) and (7) that we can obtain Bessel-Gauss solutions to the damped wave and Klein-Gordon equations. Using Refs. [10] and [7], a family of Bessel-Gauss damped wave equation solutions is

$$\Phi_n(\mathbf{r},t)=\frac{a_1}{V}J_n\left[\frac{\kappa a_1\rho}{V}\right]\times\exp\left[\frac{-\beta\rho^2}{V}\pm in\phi\frac{-i\kappa^2a_1\zeta}{4\beta V}-|b|ct+i(zp_1+ctp_2)\right] , \tag{10}$$

where b, p_1, and p_2 are given in (5), κ is a spectral parameter, a_1 is a constant, and $V=a_1+i\zeta$ [10].

Similarly, Bessel-Gauss solutions of the Klein-Gordon equation are

$$\pi_n(\mathbf{r},t)=\frac{a_1}{V}J_n\left[\frac{\kappa a_1\rho}{V}\right]\exp\left[\frac{-\beta\rho^2}{V}\pm in\phi\frac{-i\kappa^2a_1\zeta}{4\beta V}+i(zp_3+ctp_4)\right] , \tag{11}$$

where p_3 and p_4 are given by (8).

We refer to any solution to (3) as an envelope function for the solutions of the wave, damped wave, and Klein-Gordon equations given by (2), (4), and (7), respectively. Since localized solutions to three important partial differential equations are dependent upon solutions to (3), we solve (3) by choosing a dimensionless complex similarity variable and a specific variable separation ansatz that reduces (3) to an ordinary differential equation (ODE). An exact general solution to this ordinary differential equation can be found based on the confluent hypergeometric functions. The solutions of the first kind are finite at the origin while the solutions of the second kind are singular at the origin. Certain special cases of these functions reduce to localized wave solutions previously reported in the literature, but most of them are new.

We obtain a continuum of localized wave similarity solutions of (3) and also a continuum of nonlocalized wave similarity solutions of (3). A continuum parameter allows us to choose a wave with almost any transverse form in space-time. These solutions of (3) in turn give us continua of solutions to $\Box\psi(\mathbf{r},t)=0$ and Eqs. (6) and (9). Also, we find that there are an infinite number of solutions of (3) and the above partial differential equations for every value of the azimuthal dependence.

III. SOLUTION OF THE SCHRÖDINGER-LIKE EQUATION USING A COMPLEX SIMILARITY TRANSFORMATION: AZIMUTHALLY SYMMETRIC CASE

For the azimuthally symmetric case, we wish to solve

$$\left[\nabla_t^2+4i\beta\frac{\partial}{\partial\zeta}\right]f(\rho,\zeta)=0 , \tag{12}$$

with

$$\nabla_t^2=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho} . \tag{13}$$

We introduce a dimensionless complex similarity variable and attempt to reduce (12) to an ordinary differential equation [12,13]. Let this variable be given as

$$\sigma=\left[\frac{\beta}{a_1+i\zeta}\right]^{1/2}\rho , \tag{14}$$

where a_1 is a constant greater than zero. We set $f(\rho,\zeta)$ equal to a formal separation of variables,

$$f(\rho,\zeta)=F(\sigma)h(\zeta) . \tag{15}$$

Substitution of (14) and (15) and the appropriate derivatives into (12) gives

$$\frac{h(\zeta)}{(a_1+i\zeta)}\left[\frac{d^2F(\sigma)}{d\sigma^2}+\left[\frac{1}{\sigma}+2\sigma\right]\frac{dF(\sigma)}{d\sigma}\right]+4i\frac{dh(\zeta)}{d\zeta}F(\sigma)=0 . \tag{16}$$

By making a specific choice for $h(\zeta)$,

$$h(\zeta)=c_0(a_1+i\zeta)^q , \tag{17}$$

where c_0 is an arbitrary constant and q is any real number, substitution of (17) and its derivative into (16) yields the ordinary differential equation

$$F''+\left[\frac{1}{\sigma}+2\sigma\right]F'-4qF=0\ , \tag{18}$$

where $F'=dF/d\sigma$, etc.

Equation (18) can be solved for any real value of q using a standard power-series solution, but it is much more convenient to transform (18) to a known ODE form, if possible. Writing (18) as

$$\sigma F''+(1+2\sigma^2)F'-4q\sigma F=0 \tag{19}$$

and letting $x=-\sigma^2$ with F now a function of x, (19) becomes

$$i\sqrt{x}\left[\frac{d^2F}{dx^2}\left[\frac{dx}{d\sigma}\right]^2+\frac{dF}{dx}\frac{d^2x}{d\sigma^2}\right]+(1-2x)\left[\frac{dF}{dx}\frac{dx}{d\sigma}\right] -4qi\sqrt{x}F=0\ . \tag{20}$$

Since $dx/d\sigma=-2i\sqrt{x}$ and $d^2x/d\sigma^2=-2$, (20) becomes

$$x\frac{d^2F}{dx^2}+(1-x)\frac{dF}{dx}+qF=0\ . \tag{21}$$

This is a particular form of Kummer's equation and thus (18) has the general solution [14]

$$F(\sigma;q)=c_1M(-q;1;-\sigma^2)+c_2U(-q;1;-\sigma^2)\ , \tag{22}$$

where c_1 and c_2 are arbitrary constants and M and U are Kummer's functions. Returning to (15) and (17), the axisymmetric solutions of the Schrödinger-like equation in (12) are

$$f(\rho,\zeta;q)=[c_1M(-q;1;-\sigma^2)+c_2U(-q;1;-\sigma^2)] \times[a_1+i\zeta]^q \tag{23}$$

with $\sigma^2=\beta\rho^2/(a_1+i\zeta)$, $\zeta=z-ct$, and $q\in\mathbb{R}$.

The functions $f(\rho,\zeta;q)$ can be substituted into (2) to give solutions to the homogeneous wave equation, into (4) to give solutions to the damped wave equation, and into (7) to give solutions to the Klein-Gordon equation. Since q can be any real number, (23) provides a continuum of solutions of (12), which in turn provides continua of solutions of the above partial differential equations.

Throughout the paper, we will refer to $M(-q;1;-\sigma^2)$ as solutions of the first kind which are finite at $\sigma=0$ and to $U(-q;1;-\sigma^2)$ as solutions of the second kind that are singular at $\sigma=0$. For the special case where q is a negative integer, some of the solutions of the first kind have been found previously [1–3]. But solutions of the first kind containing all other values of q have not been reported while the solutions of the second kind have not been reported in the context of localized waves [19].

Since the confluent hypergeometric functions are difficult to visualize in general, we now consider some interesting special cases of (22) and (23).

1. $q=-1$

When $q=-1$, (22) becomes

$$F(\sigma;-1)=e^{-\sigma^2}[c_1+c_2\Gamma(0;-\sigma^2)]\ , \tag{24}$$

where $\Gamma(y;x)$ is the complementary incomplete gamma function [15]. Thus the solution to (23) in this case is

$$f(\rho,\zeta;-1)=e^{-\sigma^2}[c_1+c_2\Gamma(0;-\sigma^2)][a_1+i\zeta]^{-1}\ . \tag{25}$$

Choosing the second constant c_2 equal to zero, (25) becomes

$$f^{(1)}(\rho,\zeta;-1)=\frac{c_1e^{-\beta\rho^2/a_1+i\zeta}}{(a_1+i\zeta)}\ . \tag{26}$$

(The superscript 1 refers to a solution of the first kind.)

Choosing $c_1=1/4\pi i$ and substituting (26) into (2) gives

$$\psi(\mathbf{r},t)=\frac{1}{4\pi i}\frac{e^{-\beta\rho^2/V}}{V}e^{i\beta(z+ct)}\ , \tag{27}$$

with $V=a_1+i(z-ct)$, which is the fundamental Gaussian localized wave solution of the homogeneous wave equation [7–9,16]. Substitution of (26) into (4) and (7) gives fundamental Gaussian localized wave solutions of the damped wave and Klein-Gordon equations, respectively. Plots of these solutions appear in Ref. [8].

Choosing the first constant c_1 equal to zero in (25) gives

$$f^{(2)}(\rho,\zeta;-1)=\frac{c_2e^{-\beta\rho^2/(a_1+i\zeta)}\Gamma[0;-\beta\rho^2/(a_1+i\zeta)]}{(a_1+i\zeta)} \tag{28}$$

Equation (28) is a solution of the second kind and is singular at $\rho=0$. Choosing $c_2=1/4\pi i$, (28) can be written as

$$f^{(2)}(\rho,\zeta;-1)=f^{(1)}(\rho,\zeta;-1)\Gamma[0;-\beta\rho^2/(a_1+i\zeta)]\ . \tag{29}$$

Substituting (28) into (2) gives

$$\psi(\mathbf{r},t)=\frac{1}{4\pi i}\frac{e^{-\beta\rho^2/V}}{V}\Gamma(0;-\beta\rho^2/V)e^{i\beta(z+ct)}\ . \tag{30}$$

Equation (30) is the singular counterpart to (27) and is also a solution of the homogeneous wave equation. Substitution of (28) into (4) and (7) again produces singular (at $\rho=0$) solutions of the damped wave and Klein-Gordon equations, respectively.

2. $q=0$

This case must be considered separately from the nonzero values of q. Returning to (18) with $q=0$, we have

$$F''(\sigma;0)+\left[\frac{1}{\sigma}+2\sigma\right]F'(\sigma;0)=0\ . \tag{31}$$

As usual, by setting $F'(\sigma;0)=u(\sigma)$, (31) becomes

$$u'+\left[\frac{1}{\sigma}+2\sigma\right]u=0 \tag{32}$$

or

$$u(\sigma)=u_0 e^{-\sigma^2}/\sigma \;, \qquad (33)$$

where u_0 is a constant of integration.

Thus integrating again gives

$$F(\sigma;0)=c_2\mathrm{Ei}(-\sigma^2)=-c_2\Gamma(0;\sigma^2) \qquad (34)$$

when the second constant of integration is taken as the Euler constant.

Since this is a singular solution at $\sigma=0$, and $M(0;1;-\sigma^2)=1$, the general solution to (31) is

$$F(\sigma;0)=c_1+c_2\mathrm{Ei}(-\sigma^2) \;. \qquad (35)$$

Since $h(\zeta)=c_0$ in this case (we take $c_0=1$ for convenience),

$$f(\rho,\zeta;0)=c_1+c_2\mathrm{Ei}[-\beta\rho^2/(a_1+i\zeta)] \;. \qquad (36)$$

Throughout the remainder of the paper, it is understood that any $f(\rho,\zeta;q)$ such as (36) is an envelope function and produces solutions of the homogeneous wave, damped wave, and Klein-Gordon equations when using (2), (4), and (7), respectively.

3. q = 1

When $q=1$, the general solution to (18) is

$$F(\sigma;1)=c_1L_1(-\sigma^2)+c_2[e^{-\sigma^2}+L_1(-\sigma^2)\mathrm{Ei}(-\sigma^2)] \;, \qquad (37)$$

where L_1 is a Laguerre polynomial. Then

$$f(\rho,\zeta;1)=\{c_1L_1(-\sigma^2)+c_2[e^{-\sigma^2}+L_1(-\sigma^2)\mathrm{Ei}(-\sigma^2)]\}\times[a_1+i\zeta] \qquad (38)$$

satisfies (12) in this case.

In general, we find that the solutions of the first kind $F^{(1)}(\sigma;q)=c_1M(-q;1;-\sigma^2)$ can be written more familiarly in terms of Laguerre functions as

$$F^{(1)}(\sigma;q)=c_1\begin{cases} L_q(-\sigma^2), & q\geq 0 \\ e^{-\sigma^2}L_{(-1-q)}(\sigma^2), & q<0 \;. \end{cases} \qquad (39)$$

When q is an integer and less than zero, the solutions in (39) have been reported previously [2]. The remaining solutions of the first kind in (39) are new as well as all the solutions of the second kind,

$$F^{(2)}(\sigma;q)=c_2U(-q;1;-\sigma^2) \;. \qquad (40)$$

A. Similarity

The $F(\sigma;q)$ functions are similarity solutions of (18). For a given value of q, this family of functions is similar in the following sense: for any given fixed value of $\zeta=\zeta_0=z_0-ct_0$, and a given value of a_1, we also have a fixed complex variance $1/V_0=1/[a_1+i(z_0-ct_0)]$, Refs. [16,17]. The variable σ^2 can be rewritten as

$$\sigma^2=\frac{\beta\rho^2}{a_1^2+\zeta^2}(a_1-i\zeta) \;. \qquad (41)$$

At $\zeta=\zeta_0$,

$$\sigma=\left[\frac{\beta}{a_1^2+\zeta_0^2}\right]^{1/2}(a_1-i\zeta_0)^{1/2}\rho \;. \qquad (42)$$

Thus if a_1 and ζ_0 are changed by identical amounts, the variable σ can be replaced by (42), which is simply a scalar multiple of ρ in both its real and imaginary parts. Because the solutions of (12), $f(\rho,\zeta;q)$, are given by $f(\rho,\zeta;q)=F(\sigma;q)h(\zeta)$, for given values of $\zeta=\zeta_0$ and a_1, we have $h(\zeta_0)=c_0(a_1+i\zeta_0)^q$, which is constant for fixed

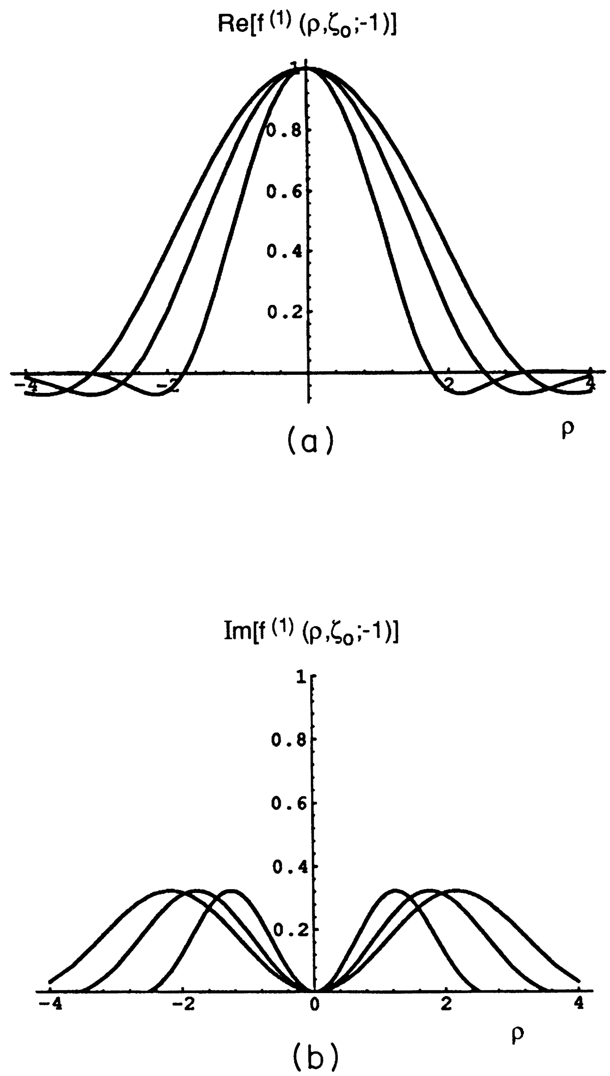

FIG. 1. (a) Similarity profiles of $\mathrm{Re}[f^{(1)}(\rho,\zeta_0;-1)]$ with $\beta=1$, and $a_1+i\zeta_0=1+i$, $2+2i$, and $3+3i$. (b) Similarity profiles of $\mathrm{Im}[f^{(1)}(\rho,\zeta_0;-1)]$ with $\beta=1$, and $a_1+i\zeta_0=1+i$, $2+2i$, and $3+3i$.

q. Thus the envelope functions $f(\rho,\zeta;q)$ are also similarity solutions in the same sense as the $F(\sigma;q)$.

If we choose $c_1 = c_0^{-1}(a_1+i\zeta_0)^{-q}$, then for fixed ζ_0

$$f^{(1)}(\rho,\zeta_0;q) = M\left[-q;1;\frac{-\beta\rho^2}{a_1+i\zeta_0}\right], \tag{43a}$$

and if $c_2 = c_1$ above,

$$f^{(2)}(\rho,\zeta_0;q) = U\left[-q;1;\frac{-\beta\rho^2}{a_1+i\zeta_0}\right]. \tag{43b}$$

Figure 1 shows plots of the real and imaginary parts of (43a) versus ρ for values of V_0 equal to $1+i$, $2+2i$, and $3+3i$ and $q=-1$. Their similarity is obvious. Figure 2 shows the same plots for (43b). Figure 3 shows what happens when ζ_0 remains fixed and a_1 only is increased. In Fig. 3, we have plotted the real and imaginary parts of (43a) with $q=-1$, $\zeta_0=1$, $\beta=1$, and $a_1=1,2,5$. If a_1 and ζ_0 do not change by identical amounts, we lose the similarity of the profiles. Both Figs. 1 and 2 show that the transverse profiles for appropriate values of a_1 and ζ_0 differ only by coordinate scale changes.

B. Localized versus nonlocalized waves

Returning to (23), we find that when q is negative, (23) provides a continuum of axisymmetric localized wave solutions to (12). Figure 4 is a three-dimensional plot of

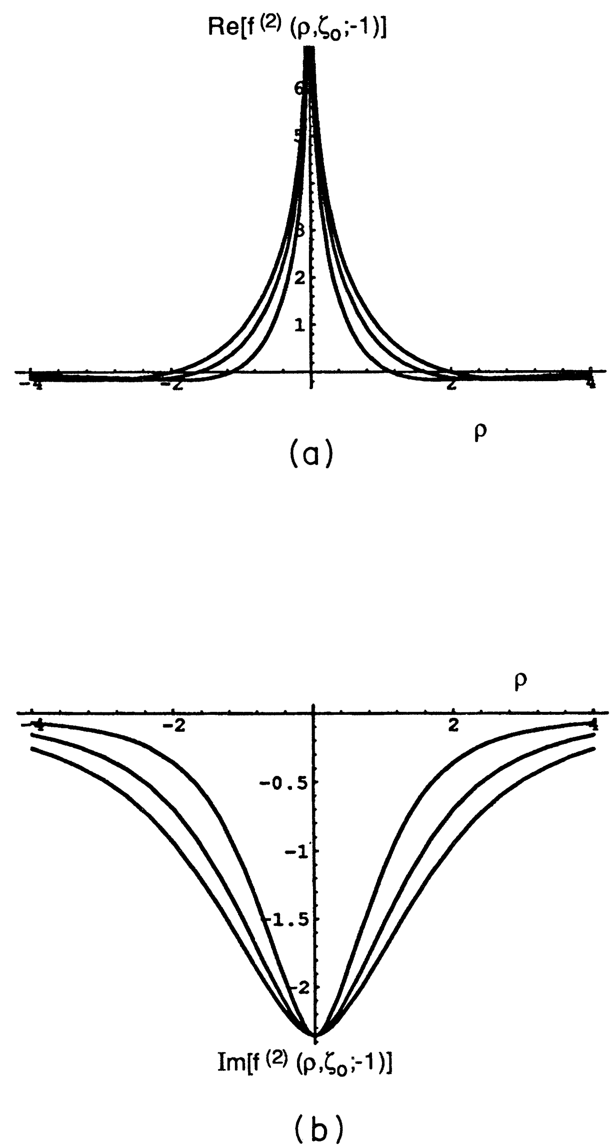

FIG. 2. (a) Similarity profiles of $\mathrm{Re}[f^{(2)}(\rho,\zeta_0;-1)]$ with $\beta=1$, and $a_1+i\zeta_0=1+i$, $2+2i$, and $3+3i$. (b) Similarity profiles of $\mathrm{Im}[f^{(2)}(\rho,\zeta_0;-1)]$ with $\beta=1$, and $a_1+i\zeta_0=1+i$, $2+2i$, and $3+3i$.

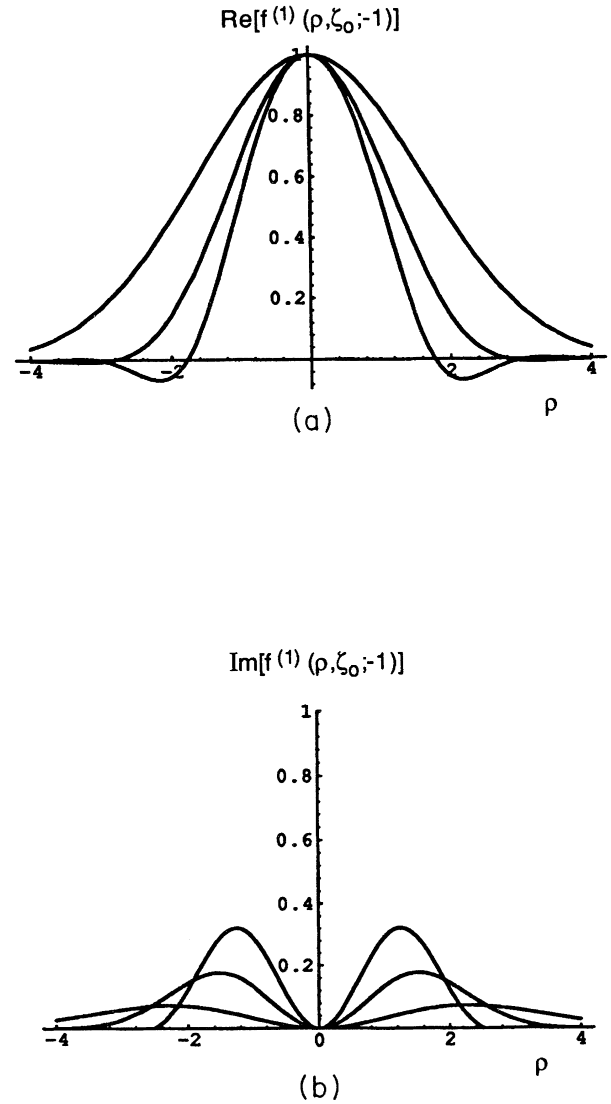

FIG. 3 (a) Profiles of $\mathrm{Re}[f^{(1)}(\rho,\zeta_0;-1)]$ that do not exhibit similarity with $\beta=1$, and $a_1+i\zeta_0=1+i$, $2+i$, and $5+i$. (b) Profiles of $\mathrm{Im}[f^{(1)}(\rho,\zeta_0;-1)]$ that do not exhibit similarity with $\beta=1$, and $a_1+i\zeta_0=1+i$, $2+i$, and $5+i$.

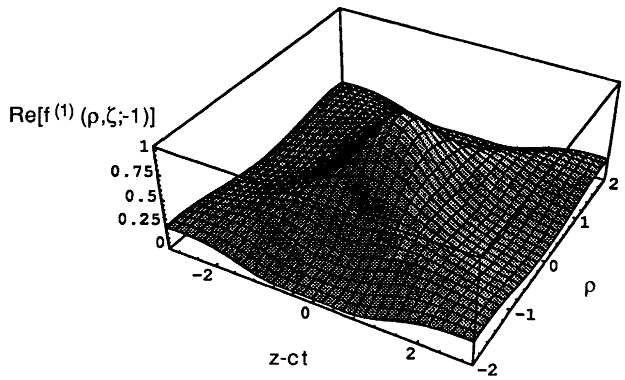

FIG. 4. Three-dimensional plot of $\mathrm{Re}[f^{(1)}(\rho,\zeta;-1)]$ vs $\zeta=z-ct$ and ρ with $\beta=1$, $a_1=1$.

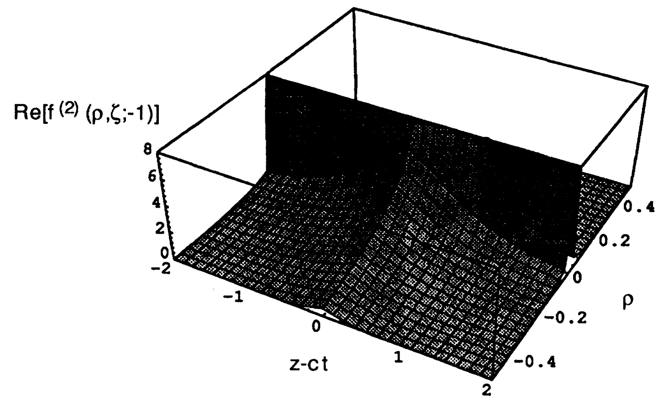

FIG. 6. Three-dimensional plot of $\mathrm{Re}[f^{(2)}(\rho,\zeta;-1)]$ vs $\zeta=z-ct$ and ρ with $\beta=1$, $a_1=1$.

the real part of $f^{(1)}(\rho,\zeta;-1)$ while Fig. 5 is a plot of the real part of $f^{(1)}(\rho,\zeta;-3)$. Figure 4 is the real part of the envelope of the fundamental Gaussian solutions of the wave, damped wave, and Klein-Gordon equations. Figure 5 is the real part of the envelope of a higher-order member of this family. Figure 6 is a plot of the real part of $f^{(2)}(\rho,\zeta;-1)$ while Fig. 7 is a plot of $f^{(2)}(\rho,\zeta;-3)$. As q becomes zero or positive, we have a continuum of nonlocalized envelope solutions of (12) that are shown in Figs. 8 and 9. These nonlocalized envelopes in turn provide nonlocalized solutions of the homogeneous wave, damped wave, and Klein-Gordon equations but are presently of little physical interest.

IV. SOLUTION OF THE SCHRÖDINGER-LIKE EQUATION USING A COMPLEX SIMILARITY TRANSFORMATION: GENERAL CASE

Assuming a ϕ dependence of the form $e^{i\nu\phi}$ and letting $w(\rho,\phi,\zeta)=g(\rho,\zeta)e^{i\nu\phi}$, we have

$$\left[\nabla_t^2+4i\beta\frac{\partial}{\partial\zeta}\right]w(\rho,\phi,\zeta)=0\ , \tag{44}$$

and the transverse Laplacian becomes

$$\nabla_t^2=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{\nu^2}{\rho^2}\ . \tag{45}$$

The homogeneous wave, damped wave, and Klein-Gordon equations are now satisfied by using $w(\rho,\phi,\zeta)$ in place of $f(\rho,\zeta)$ in (2), (4), and (7), respectively.

Again, choosing (14) as the similarity variable and setting

$$w(\rho,\phi,\zeta)=G(\sigma)h(\zeta)e^{i\nu\phi}\ , \tag{46}$$

where $h(\zeta)$ is given by (15), we derive an ordinary differential equation in the similarity variable,

$$G''+\left[\frac{1}{\sigma}+2\sigma\right]G'-\left[\frac{\nu^2}{\sigma^2}+4q\right]G=0\ , \tag{47}$$

with $G'=dG/d\sigma$, etc. Naturally (47) reduces to (18) when $\nu=0$.

We can show that (47) is a specific form of the general confluent hypergeometric equation [14], and we find that the general solution of (47) is

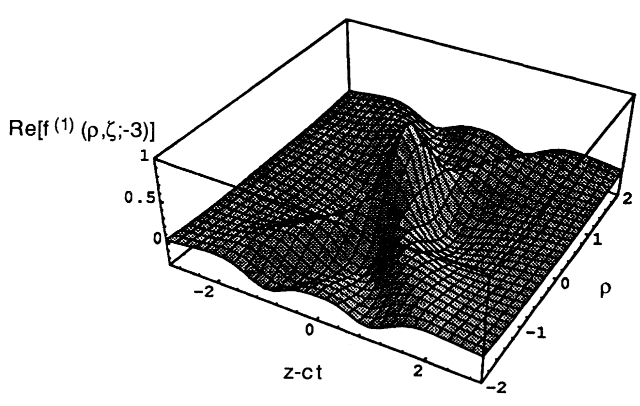

FIG. 5. Three-dimensional plot of $\mathrm{Re}[f^{(1)}(\rho,\zeta;-3)]$ vs $\zeta=z-ct$ and ρ with $\beta=1$, $a_1=1$.

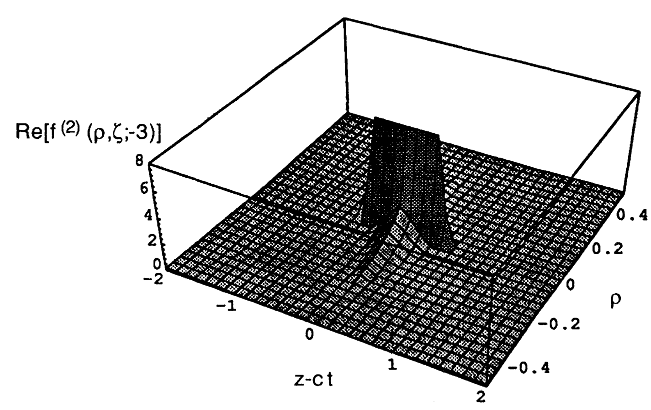

FIG. 7. Three-dimensional plot of $\mathrm{Re}[f^{(2)}(\rho,\zeta;-3)]$ vs $\zeta=z-ct$ and ρ with $\beta=1$, $a_1=1$.

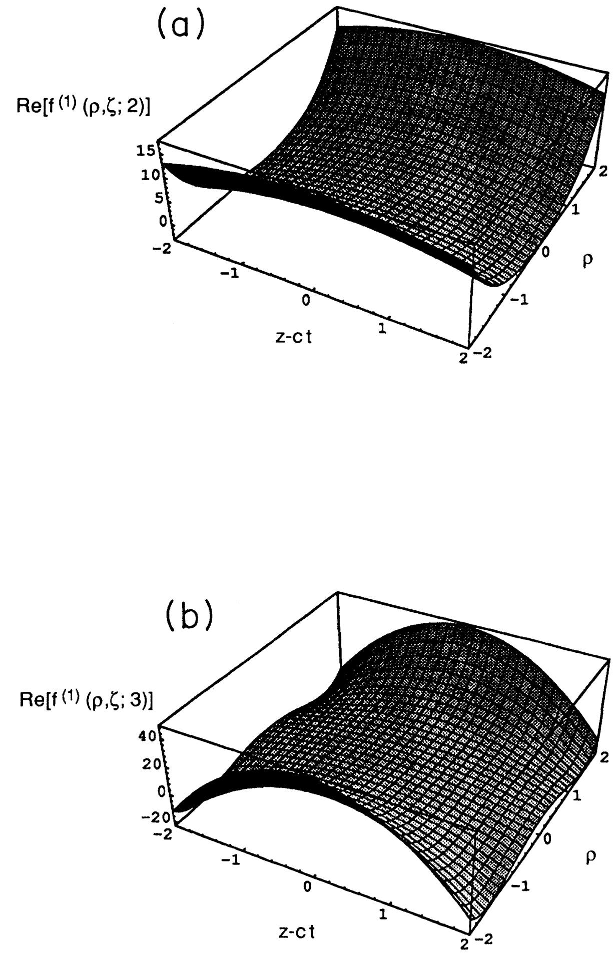

FIG. 8. (a) Three-dimensional plot of $\mathrm{Re}[f^{(1)}(\rho,\zeta;2)]$ which is nonlocalized (with $\beta=1$, $a_1=1$). (b) Three-dimensional plot of $\mathrm{Re}[f^{(1)}(\rho,\zeta;3)]$ which is nonlocalized (with $\beta=1$, $a_1=1$).

$$G(\sigma;q,\nu)=c_1\sigma^\nu M\left[\frac{\nu}{2}-q;\nu+1;-\sigma^2\right]+c_2\sigma^\nu U\left[\frac{\nu}{2}-q;\nu+1;-\sigma^2\right], \tag{48}$$

with $\nu+1\neq -N$; $N=0,1,2,\ldots$, and c_1 and c_2 are arbitrary constants. Using the notation in Sec. III, $G(\sigma;q,0)=F(\sigma;q)$. As in the axisymmetric case, the $M[(\nu/2)-q;\nu+1;-\sigma^2]$ are finite, well-behaved solutions of the first kind at $\rho=0$, while the $U[(\nu/2)-q;\nu+1;-\sigma^2]$ are singular (at $\rho=0$) solutions of the second kind. Thus solutions of (44) are

$$w(\rho,\phi,\zeta;q,\nu)=\sigma^\nu\left[c_1M\left[\frac{\nu}{2}-q;\nu+1;-\sigma^2\right]+c_2U\left[\frac{\nu}{2}-q;\nu+1;-\sigma^2\right]\right]\times[a_1+i\zeta]^q e^{i\nu\phi}\,,\quad \nu+1\neq -N,\ N=0,1,2,\ldots\,. \tag{49}$$

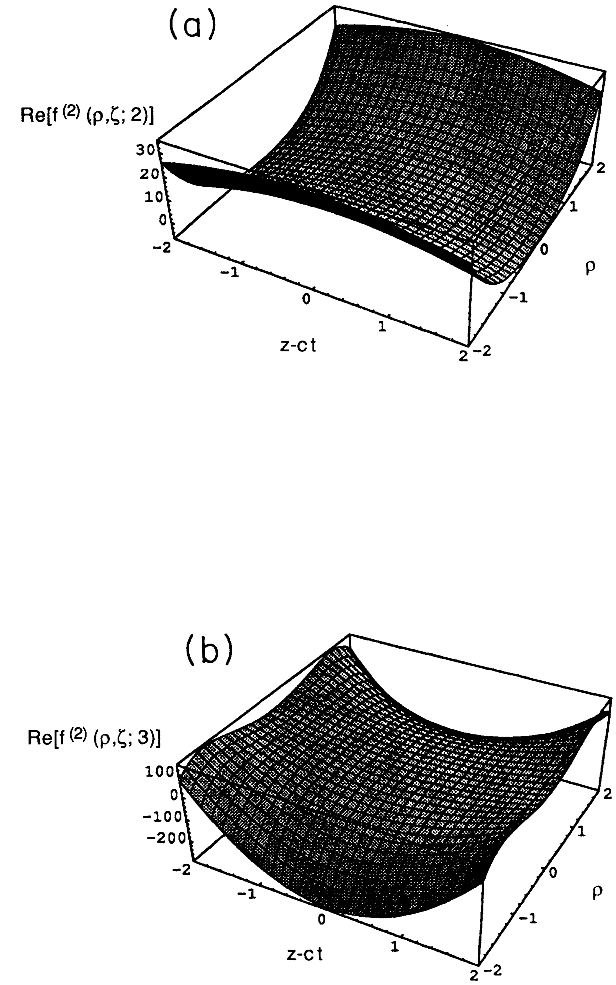

FIG. 9. (a) Three-dimensional plot of $\mathrm{Re}[f^{(2)}(\rho,\zeta;2)]$ which is nonlocalized (with $\beta=1$, $a_1=1$). (b) Three-dimensional plot of $\mathrm{Re}[f^{(2)}(\rho,\zeta;3)]$ which is nonlocalized (with $\beta=1$, $a_1=1$).

In terms of satisfying (44), ν and $q\in\mathbb{R}$ and can take on any values except ν cannot be a negative integer.

Again, it is informative to consider some of the special cases of (48) and (49).

1. $q=-\frac{3}{2}$, $\nu=1$

Using (48) with $c_2=0$, we have

$$G^{(1)}(\sigma;-\tfrac{3}{2},1)=c_1\sigma M(2;2;-\sigma^2)=c_1\sigma e^{-\sigma^2}\,. \tag{50}$$

Then (49) becomes

$$w^{(1)}(\rho,\phi,\zeta;-\tfrac{3}{2},1)=\frac{c_1\beta^{1/2}\rho e^{-\beta\rho^2/(a_1+i\zeta)}e^{i\phi}}{(a_1+i\zeta)^2}\,. \tag{51}$$

Substituting (51) into (2) and choosing $c_1=\beta^{-1/2}$,

$$\psi^{(1)}(\mathbf{r},t;-\tfrac{3}{2},1)=\frac{\rho\exp[-\beta\rho^2/(a_1+i\zeta)+i\phi+i\beta(z+ct)]}{(a_1+i\zeta)^2}\,, \tag{52}$$

which is the complex conjugate of Hillion's solution ψ_1, Eq. (2) of Ref. [5].

Using (48) with $c_1=0$, we have

$$G^{(2)}(\sigma;-\tfrac{3}{2},1)=c_2\sigma U(2;2;-\sigma^2)=c_2\sigma e^{-\sigma^2}\Gamma(-1;-\sigma^2)\ . \tag{53}$$

Then (49) becomes (with $c_2=\beta^{-1/2}$)

$$w^{(2)}(\rho,\phi,\zeta;-\tfrac{3}{2},1)=\frac{\rho}{(a_1+i\zeta)^2}\exp[-\beta\rho^2/(a_1+i\zeta)+i\phi]\times\Gamma[-1;-\beta\rho^2/(a_1+i\zeta)]\ . \tag{54}$$

This is the singular solution of the second kind for this case as compared to (51). The corresponding wave equation solution is

$$\psi^{(2)}(\mathbf{r},t;-\tfrac{3}{2},1)=w^{(2)}(\rho,\phi,\zeta;-\tfrac{3}{2},1)e^{i\beta(z+ct)}\ , \tag{55}$$

as usual.

2. $q=-\frac{1}{2}(a+1)$, $\nu=a-1$

In general, by assuming that a is any positive integer greater than one, we can obtain all of Hillion's solutions [5]. In this case,

$$G^{(1)}(\sigma;-\tfrac{1}{2}(a+1),a-1)=c_1\sigma^{a-1}M(a;a;-\sigma^2)=c_1\sigma^{a-1}e^{-\sigma^2}\ . \tag{56}$$

If $c_1=\beta^{-1/2(a-1)}$, the solution to (44) in this case is

$$w^{(1)}(\rho,\phi,\zeta;-\tfrac{1}{2}(a+1),a-1)=\frac{\rho^{a-1}\exp[-\beta\rho^2/(a_1+i\zeta)+i(a-1)\phi]}{(a_1+i\zeta)^a}\ . \tag{57}$$

Using (2), if a in (57) is an integer, we obtain the solutions in Refs. [3] and [5]. If a is a noninteger in (57) and subsequently used in (2), then (57) is the noninteger extension of Hillion's and Sezginer's solutions. While having a noninteger value may not be applicable to solutions in free space, it may be very useful for certain types of boundary-value problems.

The solutions of the second kind for this case are

$$G^{(2)}(\sigma;-\tfrac{1}{2}(a+1),a-1)=c_2\sigma^{a-1}U(a;a;-\sigma^2) \tag{58}$$

or

$$G^{(2)}(\sigma;-\tfrac{1}{2}(a+1),a-1)=c_2\sigma^{a-1}e^{-\sigma^2}\Gamma(1-a;-\sigma^2)\ . \tag{59}$$

If $c_2=\beta^{-1/2(a-1)}$, then

$$w^{(2)}(\rho,\phi,\zeta;-\tfrac{1}{2}(a+1),a-1)=w^{(1)}(\rho,\phi,\zeta;-\tfrac{1}{2}(a+1),a-1)\Gamma\left[1-a;\frac{-\beta\rho^2}{a_1+i\zeta}\right]\ . \tag{60}$$

3. $q=-\frac{1}{2}$, $\nu=2a-1$

In this case, the solutions of the first kind given by (48) and (49) are [if $c_1=(-1/4)^{a-1/2}/\Gamma(a+1/2)$]

$$G^{(1)}(\sigma;-\tfrac{1}{2},2a-1)=e^{-\sigma^2/2}I_{a-1/2}\left[\frac{-\sigma^2}{2}\right] \tag{61}$$

and

$$w^{(1)}(\rho,\phi,\zeta;-\tfrac{1}{2},2a-1)=\frac{e^{-\beta\rho^2/2(a_1+i\zeta)}I_{a-1/2}\left[\frac{-\beta\rho^2}{2(a_1+i\zeta)}\right]e^{i(2a-1)\phi}}{(a_1+i\zeta)^{1/2}}\ , \tag{62}$$

where a can be any real number (provided ν is not a negative integer) and $I_{a-1/2}$ is a modified Bessel function.

The solutions of the second kind are [if $c_2=\sqrt{\pi}(-1)^{a-1/2}$]

$$G^{(2)}(\sigma;-\tfrac{1}{2},2a-1)=e^{-\sigma^2/2}K_{a-1/2}\left[\frac{-\sigma^2}{2}\right] \tag{63}$$

and

$$w^{(2)}(\rho,\phi,\zeta;-\tfrac{1}{2},2a-1)=\frac{e^{-\beta\rho^2/2(a_1+i\zeta)}K_{a-1/2}\left[\frac{-\beta\rho^2}{2(a_1+i\zeta)}\right]e^{i(2a-1)\phi}}{(a_1+i\zeta)^{1/2}}\ , \tag{64}$$

where $K_{a-1/2}$ is a modified Bessel function. If a is chosen to be $\frac{1}{2}$, (64) becomes

$$w^{(2)}(\rho,\phi,\zeta;-\tfrac{1}{2},0)=\frac{e^{-\beta\rho^2/2(a_1+i\zeta)}K_0\left[\dfrac{-\beta\rho^2}{2(a_1+i\zeta)}\right]}{(a_1+i\zeta)^{1/2}}, \quad (65)$$

and is very similar to one of Hillion's solutions given by Eq. (20) in Ref. [18], except for some constants. Also with $a=\frac{1}{2}$, (62) is very similar to Eq. (22) in Ref. [18], again except for some constants.

Obviously many more special cases of (48) and (49) may be found which can be used in (2), (4), and (7) to satisfy the wave, damped wave, and Klein-Gordon equations. Just as in the axisymmetric case, when $(\nu/2)-q$ is positive, a continuum of localized solutions to (44) results. When $(\nu/2)-q$ is negative, a continuum of nonlocalized solutions results. Also we note that for every value of angular dependence, there are a corresponding infinite number of solutions to (44), the wave, damped wave, and Klein-Gordon equations.

V. CONCLUSIONS

In this paper, we have solved a Schrödinger-like equation using a complex similarity transformation. This enabled the partial differential equation to be reduced to an ordinary differential equation to which general solutions were determined for both the axisymmetric case and the case of general azimuthal dependence $e^{i\nu\phi}$. These general solutions contained real parameters allowing continua of solutions that satisfied the original equation. Using the forms (2), (4), and (7), these continua of solutions to the Schrödinger-like equation provide continua of solutions to the homogeneous wave, damped wave, and Klein-Gordon equations, respectively. Based on the values of the parameters q and ν, we obtained both localized and nonlocalized solutions to the above partial differential equations.

It is worth noting that $f(\rho,\zeta;q)$ and $w(\rho,\phi,\zeta;q,\nu)$ are also similarity solutions of the true free-particle two-dimensional Schrödinger equation if the formal correspondences $(a_1+i\zeta)\leftrightarrow it$ and $\beta\leftrightarrow m/2\hbar$ are used. Similarly f and w are solutions to the diffusion equation if $(a_1+i\zeta)\leftrightarrow t$ and $\beta\leftrightarrow 1/4D$, with D as the diffusion coefficient.

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